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# PLANAR ELASTIC PROBLEM FOR AN ORTHOTROPIC PLANE WITH A SLIT UNDER EDGE CONTACT CONDITIONS OF THE TYPE OF VISCOUS FRICTION

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*A system of hypersingular equations for the title problem is constructed. Qualitative properties of the solution of this system are discussed.*

**Introduction.** Previously [1], we studied the problem of longitudinal shear of a plane having a slit filled with a material with low shear compliance. In the present paper, a similar elastic problem is considered. Here, in contrast to the antiplane problem, two versions of conjugation conditions at the edges of the slit exist: 1) the shear stress is proportional to the jump in tangential displacement; 2) the normal stress is proportional to the jump in normal displacement. Conjugation conditions of this kind arise in the theory of elasticity in solving the problem of contact of two domains with a thin interlayer whose coefficient of elasticity is much larger than those of the embedding medium (see, for instance, [2]). The conjugation problems considered are reduced to systems of hypersingular equations. It is shown that they coincide with accuracy to coefficients with the system derived in [1] and admit the same analysis.

**1.** We consider the first version. We write the Hooke's law for an orthotropic material as

$$\sigma_{11} = d_{11}\frac{\partial u_1}{\partial x} + d_{12}\frac{\partial u_2}{\partial y}, \qquad \sigma_{22} = d_{12}\frac{\partial u_1}{\partial x} + d_{22}\frac{\partial u_2}{\partial y}, \qquad \sigma_{12} = \frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x}.$$

It is assumed that the matrix of elastic constants is positive-definite: $d_{11} > 0$, $d_{22} > 0$, and $d_{11}d_{22} - d_{12}^2 > 0$. The stresses and coefficients in the Hooke's law are dimensionless and are normalized to the shear modulus and $u_1$ and $u_2$ are the displacements. Let the slit lie on the straight line $y = 0$ and occupy the interval $(-a, a)$ on the $x$ axis, $R^2_-$ be the lower half-plane $y < 0$, and $R^2_+$ be the upper half-plane $y > 0$. We specify conjugation conditions for the half-planes in the form

$$[u_2] = 0, \qquad [\sigma_{22}] = 0, \qquad y = \pm 0; \tag{1}$$

$$[u_1] = 0, \qquad [\sigma_{12}] = 0, \qquad y = \pm 0, \qquad |x| > a; \tag{2}$$

$$\sigma_{12}(x, +0) = k[u_1](x, +0) + f(x), \qquad |x| \leqslant a; \tag{3}$$

$$\sigma_{12}(x, -0) = k[u_1](x, -0), \qquad |x| \leqslant a. \tag{4}$$

Here the function $f(x)$ is known and it models the action of surface forces on the slit; square brackets designate a jump in the function. The coefficient $k$ is specified and positive, and it is called the viscosity coefficient.

We explain in detail the physical meaning of conditions (3) and (4). From the physical viewpoint, it is assumed that the slit is filled with an elastic material with a low Young's modulus, the thickness of the slit and the Young's modulus being comparable in order of magnitude. For the Laplace equation (steady heat conduction), a similar problem was studied by Sanchez-Palencia [3]. From the mechanical viewpoint, conditions (3) and (4) imply that the variation of the solution along the $x$ coordinate is negligible compared to that along the $y$ coordinate. However, this is not a unique situation where similar conditions can arise. For

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example, if we assume that the slit is filled with a large number of microslits and perform averaging over the number of microslits, we also arrive at conditions of type (3) and (4).

We reduce the elastic problem with boundary conditions (1)–(4) to a system of integral equations using explicit elastic solutions for the displacements specified at the boundary for the upper and lower half-planes, respectively. The functions specified on the boundary will be called densities. We use conjugation conditions (1)–(4) to obtain a system of integral equations for the densities. We introduce the following integral operators (analogs of the potentials of the single and double layers for the Laplace equation):

$$\mathcal{K}_1(f,\lambda y)=\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{f(t)\lambda y}{(x-t)^2+\lambda^2y^2}\,dt,\quad \mathcal{K}_2(f,\lambda y)=\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{f(t)(x-t)}{(x-t)^2+\lambda^2y^2}\,dt.$$

Let $\gamma_1=(1+d_{12})/(d_{11}-\lambda_1^2)$ and $\gamma_2=(1+d_{12})/(d_{11}-\lambda_2^2)$. Here $\lambda_1$ and $\lambda_2$ are positive roots of the equation $d_{22}\lambda^4-(d_{11}d_{22}-d_{12}^2-2d_{12})\lambda^2+d_{11}=0$.

Let displacements be specified for $y=+0$: $u_1(x,+0)=f_1(x)$ and $u_2(x,+0)=f_2(x)$. In this case, the elastic solution in the upper half-plane is given by the formulas

$$u_1^1(x,y)=\frac{\gamma_1}{\gamma_1-\gamma_2}\mathcal{K}_1(f_1,\lambda_1y)-\frac{\gamma_2}{\gamma_1-\gamma_2}\mathcal{K}_1(f_1,\lambda_2y)+\frac{\gamma_1\gamma_2}{\gamma_1-\gamma_2}[\mathcal{K}_2(f_2,\lambda_1y)-\mathcal{K}_2(f_2,\lambda_2y)],$$

$$u_2^1(x,y)=\frac{1}{\gamma_1-\gamma_2}[\mathcal{K}_2(f_1,\lambda_1y)-\mathcal{K}_2(f_1,\lambda_2y)+\gamma_1\mathcal{K}_1(f_2,\lambda_2y)-\gamma_2\mathcal{K}_1(f_2,\lambda_1y)].$$

We set $u_1(x,-0)=f_3(x)$ and $u_2(x,-0)=f_4(x)$. The solution in the lower half-plane has the form

$$u_1^2(x,y)=\frac{\gamma_1}{\gamma_1-\gamma_2}[-\mathcal{K}_1(f_3,\lambda_1y)+\gamma_2\mathcal{K}_2(f_4,\lambda_1y)]-\frac{\gamma_2}{\gamma_1-\gamma_2}[-\mathcal{K}_1(f_3,\lambda_2y)+\gamma_1\mathcal{K}_2(f_4,\lambda_2y)],$$

$$u_2^2(x,y)=\frac{1}{\gamma_1-\gamma_2}[\mathcal{K}_2(f_3,\lambda_1y)-\mathcal{K}_2(f_3,\lambda_2y)]+\frac{1}{\gamma_1-\gamma_2}[\gamma_2\mathcal{K}_1(f_4,\lambda_1y)-\gamma_1\mathcal{K}_2(f_4,\lambda_2y)].$$

The superscripts 1 and 2 refer to the upper and lower half-planes, respectively. It follows from (1) that $f_2=f_4$ and since $[\sigma_{22}]=0$, for any real $x$ we have

$$\int_{-\infty}^{+\infty}\frac{f_2(t)+f_4(t)}{(x-t)^2}\,dt=0.$$

Therefore, it is natural to set $f_2(t)+f_4(t)=0$, whence $f_2(t)=f_4(t)=0$. The absence of a jump in $u_1$ and $\sigma_{12}$ for $|x|>a$ results in $f_3(t)$ and $f_1(t)$ vanishing for $|t|>a$. One can easily find that

$$\frac{\partial u_1^1}{\partial y}=\frac{l}{\pi}\int_{-a}^{+a}\frac{f_1(t)}{(x-t)^2}\,dt,\qquad \frac{\partial u_1^2}{\partial y}=-\frac{l}{\pi}\int_{-a}^{+a}\frac{f_1(t)}{(x-t)^2}\,dt.$$

Here $l=(\gamma_1\lambda_1-\gamma_2\lambda_2)/(\gamma_1-\gamma_2)=d_{11}(\lambda_1+\lambda_2)/(d_{11}+\lambda_1\lambda_2)$. This leads to the following system of hypersingular equations for the densities $f_1$ and $f_3$:

$$\frac{l}{\pi}\int_{-a}^{+a}\frac{f_1(t)}{(x-t)^2}\,dt=k(f_1(x)-f_3(x))+f(x),\quad -\frac{l}{\pi}\int_{-a}^{+a}\frac{f_3(t)}{(x-t)^2}\,dt=k(f_1(x)-f_3(x)).$$

Subtracting and summing these equations, we obtain

$$\frac{l}{\pi}\int_{-a}^{+a}\frac{m(t)}{(x-t)^2}\,dt=2km(x)+f(x); \tag{5}$$

$$\frac{l}{\pi}\int_{-a}^{+a}\frac{n(t)}{(x-t)^2}\,dt=f(x), \tag{6}$$

where $m(x) = f_1(x) - f_3(x)$ and $n(x) = f_1(x) + f_3(x)$. The same system (with accuracy to the factor $l$) was constructed in [1]. The integrals in (5) and (6) should be understood as the finite part of a diverging Hadamard integral. As shown in [1], Eq. (5) can be reduced to a Fredholm integral equation of the second kind and Eq. (6) is solvable in explicit form. In accordance with the results of [1], the solutions of Eqs. (5) and (6) for $f(x) \in C^{0,\alpha}$ belong to $C^{1,\alpha}$. Moreover, $m(x)$ and $n(x)$ vanish at the ends of the interval and their derivatives can have root singularities for $x = \pm a$. Here $C^{s,\alpha}(-a, a)$ is a Banach space of functions that have $s$ continuous derivatives, the $s$th derivative satisfying the Hölder condition with index $\alpha \leqslant 1$. For small $k$, Eq. (5) is singularly perturbed and its solution can be obtained by the method of joined asymptotic expansions [4].

**2.** We consider another version of the boundary conditions at the edges of the slit. Let

$$[u_1] = 0, \quad [\sigma_{12}] = 0, \quad \sigma_{22}(x, +0) = s[u_2] + g(x), \quad \sigma_{22}(x, -0) = s[u_2], \quad |x| \leqslant a,$$
$$[u_2] = 0, \quad [\sigma_{22}] = 0, \quad |x| > a$$

for $y = \pm 0$. It is assumed that $s > 0$. Using the above solutions of the problem for the displacements in the upper and lower half-planes, we infer that $f_1 = f_3 = 0$ and $f_2(x) = f_4(x) = 0$ for $|x| > a$. Thus, we have the following system for the densities $f_1$ and $f_3$ in the interval $(-a, a)$:

$$-\frac{d_{22} l_1}{\pi} \int_{-a}^{+a} \frac{f_2(t)}{(x-t)^2}\, dt = s[f_2(x) - f_4(x)] + g(x), \quad \frac{d_{22} l_1}{\pi} \int_{-a}^{+a} \frac{f_4(t)}{(x-t)^2}\, dt = s[f_2(x) - f_4(x)].$$

Here $l_1 = \lambda_1 \lambda_2 (\lambda_1 + \lambda_2)/(d_{11} - \lambda_1 \lambda_2)$. This system differs from system (5) and (6) only by the coefficient $d_{22} l_1$.

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